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On the growth of waves in boundary layers: a non-parallel correction

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The estimation of the growth of propagating instability waves in laminar boundary layers is considered when the Reynolds number is sufficiently large for the mean flow to deviate only slightly from a truly parallel flow. An approximate solution for the linear perturbation is sought in the form of a scaled solution of the related locally parallel flow problem. The amplitude scaling is chosen so as to satisfy the full linearized perturbation equations as closely as possible by making the mean-square deviation of the remainder a minimum. By re-arranging the terms in the equations so that some of the small correction terms arising from the non-parallel mean flow are contained in the ordinary differential equation (ODE) defining the quasi-parallel flow solution, a useful simplification is obtained for the scaling function. Then a modified Orr–Sommerfeld equation defines the base solution and the differential expression for the scaling that can be integrated forms a simple conservation relation.

1. Introduction

The study of the instabilities of laminar flows and the determination of how weak disturbances can grow into turbulence has been an on-going topic for over one hundred years. Much of the early mystery and conjecture has been removed from the subject through the enormous amount of research that has been carried out, resulting in a reasonably clear picture of the underlying physical processes that take place. There are, nevertheless, still interesting and unresolved aspects of the transition process worthy of further study. Much of the early work concentrated on the problem of predicting whether or not infinitesimal disturbances would grow or decay, hence defining the flow's stability. The framework for much of this research was put in place by Reynolds (1883) and Rayleigh (1880). Rayleigh made significant progress in explaining the instabilities of certain simple parallel base flows by ignoring the effects of viscosity, assuming that the inclusion of viscosity would only act as a stabilizing influence. It turns out that the role of viscosity in generating the stresses required for instability is quite subtle and this was not properly understood until Prandtl (see Durand 1932, vol. III) gave a clear explanation showing how the wall-induced phase variations of the eigenfunction created the required Reynolds stress for energy transfer from the mean to the perturbation. Solution of even the simplest flow that included the viscous terms in the governing equation was difficult and only geometries that enabled the linearized equations to separate and reduce to ODEs could be tackled. Even then the extraction of the eigenvalues defining stability was a formidable task. Heisenberg (see Lin 1955) employed great ingenuity in formulating the asymptotic solutions for large Reynolds numbers of the Orr–Sommerfeld equation for the case of plane channel flow to obtain estimates of the lower branch of the basic instability. The more interesting problem of determining the stability of boundary layers was only tackled later by students and colleagues of Prandtl.

The boundary layer is more amenable to experimental exploration than the parallel flow in a channel. The developing study of the aerodynamics of wings was being studied in wind tunnels in various laboratories around the world and it was clearly important to make boundary layer stability predictions for these technologically important flows. The approach used by Tollmien (1929) and Schlichting (1933) was to treat the boundary layer as a 'locally parallel flow' with a streamwise velocity profile of the boundary layer at various downstream locations on the aerofoil or plate. Boundary layer growth at the Reynolds numbers where transition occurs is generally weak and the neglect of various small terms in the perturbation equations that prevented reduction to ODEs did not appear to be too severe. The resulting 'parallel-flow' approximation enabled the partial differential equation describing the wavy disturbance to be reduced to an ODE by taking Fourier transforms in the streamwise coordinate, albeit with slowly varying parameters. The task of finding eigenvalues using the methods then available was sufficiently daunting itself without considering any errors introduced by treating the mean flow as parallel. It is to Schlichting's credit that the solutions that he obtained were close to the accurate ones that can now easily be found using a digital computer. The predictions of the amplifications of oscillatory perturbations were defined in the form of integrals of these local solutions evaluated on the basis of a purely parallel mean flow.

The effect of applying the so-called 'parallel-flow' approximation that neglected the small terms in the perturbation equations was estimated by Pretsch (1941) who considered the relative magnitudes of the various terms in the full linearized equations. He concluded that the terms ignored were all at least of order $R^{-1/2}$ smaller than those retained in the parallel flow solution, where R is the local displacement Reynolds number of the boundary layer. He therefore assumed that such weak terms would only have a marginal effect on the solution. It is worth recalling that at that time there was no experimental evidence that the findings of Schlichting had any relevance to transition in boundary layers. Taylor (1939) went so far as to question the very existence of Tollmien–Schlichting waves. Only after the remarkable experiments of Schubauer & Skramstad (1948) was there general acceptance of the idea that the predicted instabilities really did occur and that the amplification of small disturbances could lead through transition to a turbulent boundary layer. The confirmation of the theoretical predictions by these experiments was so remarkable that the underlying assumptions in the theoretical model were not really questioned. For example, the theory was a stability theory involving disturbances that evolved in time whereas the observations made in the wind tunnel were of spatially growing waves. Although Schlichting (1935) did use the group velocity to link the expected physical growth to the instability calculated, there appeared to be no real understanding of the link, nor was this formulation generally used to calculate amplification factors. Also the parallel-flow approximation was used to define the local growths without understanding the fact that there was no way of properly coupling the solutions at neighbouring streamwise steps. Despite these questions, comparisons between predictions and observations were on the whole acceptable. The theory provided a description of wave growth that was not grossly in error and it could usefully be used in many flows of practical interest. The theory was also good enough to use in simple schemes for transition prediction that were based on the linear amplification of different wave modes. Calculations of the amplification showed that transition often occurred when the most amplified wave grew by roughly e^9. This empirical growth factor provided a relatively simple way of estimating the most likely transition region. Such schemes, even with all their shortcomings, are currently used in industry today.

There were, however, small differences in the predictions of the lowest critical Reynolds numbers for the existence of unstable waves and the early observations made on a flat plate in a wind tunnel. The differences between the calculated eigenvalues and those obtained experimentally by Schubauer were easily accepted at the time because of the difficulties both of making the necessary measurements and of evaluating the eigenvalues. Since then accurate numerical solutions of the Orr–Sommerfeld equation have been obtained and also new measurements have been made (Ross *et al.* 1970), but these efforts have not resolved the problem. Comparisons of the amplification rates within the neutral loop were generally more acceptable than the location of points on the neutral loop. The amplification of the two-dimensional waves from the early experiments only enabled comparisons to be made over relatively short distances before the amplitudes became so large that nonlinear factors arose. Accurate experimental measurements are very difficult to carry out at such low Reynolds number. The theory describes the growth of a single linear eigensolution, whereas in an experiment it is impossible to avoid the excitation of other modes as well. Although these higher modes are all damped they will nevertheless inevitably contaminate the data to some degree and make it difficult accurately to locate the neutral amplification points. There are also difficulties in creating the perfect Blasius boundary layer flow in the laboratory in which to carry out the experiments.

In some work involving wavepackets significant discrepancies appeared between the amplifications observed and values calculated using the parallel-flow model (Gaster 1975). The amplitudes within an impulsively driven wavepacket grow much more slowly than an isolated two-dimensional mode because of the streamwise and spanwise spreading of the wave envelope. This means that experimental data can be obtained over vast streamwise distances involving the growth of the boundary layer thickness by a factor two or more. Under these circumstances it was perhaps less surprising that the observed behaviour was not well predicted by the quasi-parallel eigenvalues. At the time there was no theory available to take proper account of the influence of boundary layer development on wave growth and an *ad hoc* correction for the boundary layer growth based on energy transfer and dissipation was formulated. This crude scaling idea was not well received. It was clear that the problem of estimating the influence of boundary layer development on wave growth needed to be tackled.

Barry & Ross (1970) considered the influence of the neglected terms in the perturbation equations. The full perturbation equation contains small terms that have streamwise spatial variation and this prevents reduction to an ODE by Fourier transformation. By only including those terms of order $1/R^{1/2}$ that *did not* contain streamwise derivatives an ODE similar to the Orr–Sommerfeld equation was created. The terms included in this modified O–S equation arose through the normal component of the mean velocity that is equated to zero in the strictly parallel flow formulation. The calculations using this formulation certainly shifted the neutral loop towards the experimental data. It was argued that the residual terms, although strictly of similar order to those retained, were in numerical terms small because of the eigenfunction normalization used. Although this argument is incorrect, because at this level of approximation there are other terms missed out that also depend on the normalization, the modified O–S equation certainly appeared to produce stability criteria that were closer to the true non-parallel values than those given by solutions of the unmodified governing equation.

Because the quasi-parallel Orr–Sommerfeld approach gave results not too far removed from experiment, it seemed reasonable to expand the full linearized perturbation equations around that basic state and seek a correction term to account for weak boundary layer growth (Gaster 1974). The formulation used an iterative approach to create a series of correcting terms to this basic solution derived from an ODE. It was clear that some scaling of the trial solution was also needed to account for the growth of the boundary layer thickness with distance downstream, but additional terms in an expansion scheme also needed to be included. The ordering of terms involved not only the transparent scaling that arose when any term contained derivatives with respect to streamwise distance, but also any additional ordering that involved knowledge of the solution structure normal to the boundary. For reasons that are no longer obvious the ordering from the structure at the critical layer was ignored, although more recently this has been shown to be important (Govindarajan & Narasimha 1999). The use of the adjoint function of the O–S equation enabled the amplitude scaling to be determined in terms of integrals through the boundary layer involving terms formed from products of the eigenfunction and the adjoint function. The amplification of any measurable quantity was thus defined to order $1/R^{1/2}$ in terms of the eigenvalue of the O–S equation, an amplitude scaling and a term arising from the variation of eigenfunction with downstream distance. The neutral loop, evaluated for the appropriate experimentally measured quantity, was certainly shifted towards the experimental data, but did not fit the measurements convincingly.

Before Gaster (1974) was published two papers by Bouthier (1972, 1973) appeared. He used a multiple-scale approach and directly obtained the equations for the amplitude scaling function. He ordered the terms purely on the basis of the streamwise behaviour and had to artificially separate the terms arising directly from the mean flow divergence and those from the O–S equation. The apparent inconsistency of including a high-order viscous term in the basic O–S equation can readily be justified if the ordering is additionally based on the internal structure, as would be done in triple-deck asymptotics. Unfortunately the computed behaviour was not interpreted correctly and did not reflect the quantities that one would measure in an experiment. The papers were, nonetheless, the first ones published that developed a mathematical theory for the prediction of the growth of instability waves in a developing boundary layer.

Nayfeh, Saric & Mook (1974) and Saric & Nayfeh (1975) also used a multiple scale approach to obtain estimates of wave growth in developing boundary layers. They obtained a correction to the amplification rates in terms of integrals of base solutions and adjoint functions identical to the form obtained by Bouthier and by Gaster. In their approach the eigenfunction variation with downstream distance was ignored in the prediction of amplification rates because their calculations showed that these factors were numerically small compared with those retained. The relative magnitude of the terms defining amplification of a physical quantity depends on the choice of normalization, but the final result from all the terms must be independent of the normalization. It is therefore quite incorrect to ignore certain terms because in the particular normalization used they are numerically small. The neutral stability loop that was obtained by Saric & Nayfeh fell virtually on top of all the existing experimental data. No mention was made of the fact that different physical quantities amplify at slightly different rates, presumably because in their calculation scheme these factors also appeared to be weak. This result is generally quoted in text books as an illustration of the power of the non-parallel theory in predicting boundary layer stability that is consistent with the observations. It is now known to be a fortuitous result arising from a numerical error in the computer codes. This spurious correlation will, no doubt, continue to be quoted in review articles and books because the agreement with experiment is so convincing.

A study by van Stijn & van de Vooren (1982) using somewhat different numerical methods completely supported the results of Gaster (1974).

All the multiple scale schemes are somewhat similar and although there are subtle variations in the methods used all the solutions derived consist of a scaled quasi-parallel solution together with a series of correction terms. The scaling function is formed by using the adjoint operator to form a solvability integral. The series of terms appear in inverse powers of the Reynolds number and constitute an asymptotic form of solution that provides insight into the solution at infinite Reynolds number. It can be argued that there is no point in using the locally parallel flow solution as the starting term of the series and that one might just as well use a triple-deck approach to form the direct asymptotic solution to the full equations of motion as has been done by Smith (1979). At infinite Reynolds numbers this is certainly true, but at the finite Reynolds numbers of interest in experiments the predictions made by triple-deck theory are not very good. It seems, therefore, better to attempt to correct a reasonably good approximate solution based on the quasi-parallel approach than to attempt a solution to the full problem *ab initio* using triple-deck methods.

Fasel & Konzlemann (1990) tackled the problem of wave growth prediction in a boundary layer directly by numerical means. They computed the spatial evolution of a wavetrain initiated at some upstream station in a Blasius boundary layer. The flow field data obtained were contaminated by higher eigensolution just as in experiments and had to be processed in much the same way to produce the relevant growth rates. The neutral loop obtained confirmed the calculations of Gaster (1974) whilst disagreeing with the existing experimental measurements.

An attempt to resolve this problem by carrying out more precise experiments was made by Klingmann *et al.* (1993). They showed that small deviations from the Blasius mean flow could easily arise on a simple flat plate with an elliptic nose that were sufficient to explain the discrepancies between the predictions and the measurements. By modifying the nose geometry to achieve a good mean flow they found wave growth values that were in excellent agreement with Gaster (1974).

Govindarajan & Narasimha (1995, 1997, 1999) re-examined the formation of approximate predictions of wave growth in boundary layers in a series of papers. They use a somewhat different coordinate system that certainly appears to simplify the equations and helps to clarify the terms neglected in the different approximations that have been made. In the first paper (1995) the formulation of the differential equations is essentially equivalent to those of Gaster, Bouthier, Nayfeh & Saric etc., but the method of solution is different. They argue that the small slowly varying right-hand side to the ODE contains a controlling parameter based on Reynolds number, and that the solution of the full system can be accomplished by including the residual inhomogeneous element in an iterative manner. In this way they not only obtain a solution containing the effects of non-parallelism found previously, but also include the contribution from the particular integral that is normally neglected. This is generally done to avoid considerable extra computation of an inhomogeneous problem, but also because the additional terms can be shown to be small and would naturally come in at the next term of the expansion. The iterative scheme that is used certainly appears to work in this instance, but such methods can be slow to converge down to machine accuracy, or may even fail to reach a solution in some cases. However, with care it appears that amplification rates can be evaluated with sufficient accuracy for all practical purposes by this scheme. They have also calculated amplification rates for pressure-gradient boundary layers of the Falkner–Skan family of flows. Govindarajan & Narasimha (1997) noted some small differences with Gaster (1974) that they attribute to the neglect of the particular integral element of the solution by Gaster referred to above.

More recently Govindarajan & Narasimha (1999) have critically examined the magnitudes of the terms in the governing equation and obtained a consistent lowest-order ODE. It was argued that this would provide approximate solutions accounting for the spatial boundary layer growth at large Reynolds numbers. Any scheme that ignores the coupling between solutions at different streamwise locations means that the overall solution is unspecified to some degree, as in the quasi-parallel method. The local solutions will be modified by the additional terms, and presumably this affects the eigenvalues obtained, but it is not clear to what extent the integrated result will be improved. The results obtained by such a method will be dependent on the type of normalization used in evaluating the eigenfunctions. It seems, therefore, inconsistent to calculate the spatial growth of any specific quantity such a velocity at the inner maximum, because the result would be incomplete and unspecified to some extent. The eigenvalue is, however, unaffected by normalization and does indeed provide a consistent measure of some overall amplification.

2. Approach

The present approach seeks a method of estimating wave growth in developing laminar boundary layers at the finite values of Reynolds number that occur in the laboratory or on aircraft wings. It seems to be appropriate to again base the solution on some small modification of an easily formed approximate solution generated by a suitable ODE. In the previous methods discussed a series of correction terms of diminishing magnitude were in principle generated, although no more that the leading term was calculated. The amplification of a given mode was then determined to order $1/R^{1/2}$ from some solvability condition, whereas the actual solution, as far as the measurable velocity fluctuations are concerned, was only obtained to zeroth order. These methods concentrate on correcting the integrated amplification of the dominant eigenmode whilst ignoring the weak scattering of higher modes and the continuous spectrum that must contaminate the overall solution to order $1/R^{1/2}$. At this level the correction from the locally parallel form of solution arises solely in the form of an amplitude scaling as a function of streamwise location. This scaling is required to enable the quasi-parallel local solutions to be properly coupled.

Instead of using adjoint operators to obtain the amplitude scaling function, here an amplification parameter that gives the best fit to the linearized equations of motion at each downstream location will be chosen. This will enable a solution to be found that satisfies the full equations of motion, including the inhomogeneous terms that were ignored in the adjoint approach, as closely as possible in a mean-square sense. The solution will consist of a base term, generated by an ODE at each streamwise station, modified by some amplitude scaling function so that the magnitude of the residual terms in the equation are minimized. No additional correction terms are sought.

The following constraints are to be imposed on the solution:

(i) The initial level of approximation will be obtained by a solving an ordinary differential equation formed from the linearized Navier–Stokes equations.

(ii) Corrections are sought that *minimize*, in a mean-square sense, the terms neglected in the governing equation. Because the streamwise development of the mean flow is weak and the scaling correction factor, $A(\xi)$, is slowly varying higher derivatives will be ignored as well as products of first derivatives in ξ.

(iii) The current work focuses solely on the zero-pressure-gradient Blasius base flow, but there should be no difficulty in extending the scheme to more general situations, or to three-dimensional disturbances.

3. Analysis

The x and y coordinates are in the stream and normal directions respectively. A modal disturbance is deemed to exist at a distance x_o from the leading edge and the development of that disturbance is sought downstream. It is convenient at this stage to use the boundary layer coordinates (ξ, η) and the normalizations used in Gaster (1974):

$$\xi = \frac{x}{x_0}, \qquad \eta = y\left(\frac{U}{\nu x}\right)^{1/2}. \tag{1}$$

U is the free-stream velocity, ν the viscosity and the Reynolds number at x_0 is $R = Ux_0/\nu$. Then

$$\eta = \frac{U}{\nu}\frac{y}{R^{1/2}\,\xi^{1/2}}. \tag{2}$$

The mean-flow streamfunction is defined by

$$\psi = \nu\xi^{1/2}R^{1/2}f(\eta), \tag{3}$$

where for the flat-plate Blasius boundary layer $f(\eta)$ is given by

$$f''' + \tfrac{1}{2}ff'' = 0. \tag{4}$$

An initial wavy disturbance of frequency Ω exists at x_0.

$$\varpi = \frac{\Omega\, l_0}{U}. \tag{5}$$

It is convenient to use a length scale $l = \sqrt{(x\nu/U)}$ that does not involve any numerical factors. Final results will be scaled in the conventional manner using displacement thickness. l_0 is the length scale at x_0 and ϖ is therefore a constant.

A solution of the perturbation streamfunction of the linearized Navier–Stokes equation is now sought in the form

$$A(\xi)\,\psi(\xi, \eta, t), \tag{6}$$

where $A(\xi)$ is the amplitude scaling needed to cater approximately for the development of the boundary layer and $\psi(\xi, \eta, t)$ is the perturbation streamfunction given by a solution of the local parallel flow perturbation defined by the O–S, or some other related ODE.

Using the non-dimensional parameters appropriate to the boundary layer the quasi-parallel flow solution can be written as

$$\psi = \phi(\xi, \eta)\,e^{iQ}, \tag{7}$$

where

$$Q = R^{1/2}\int_1^{\xi}\frac{\alpha(\xi)}{\xi^{1/2}}\,d\xi - \varpi t. \tag{8}$$

On substituting into the linearized Navier–Stokes equation, ignoring terms that are quadratic or higher in $\partial/\partial\xi$ we get

$$\text{ODE}\,[\phi] = F_0 + G \times F_1, \tag{9}$$

where

$$G = \frac{\xi}{A(\xi)}\frac{\mathrm{d}A(\xi)}{\mathrm{d}\xi} \tag{10}$$

and F_0 and F_1 contain the remaining terms that do not satisfy the ODE. The terms on the right-hand side are small compared with those on the left-hand side by a factor of magnitude $1/R^{1/2}$. We have an unknown amplitude scaling factor $A(\xi)$ that can be chosen so that the right-hand side is made as small as possible at all values of ξ. The overall solution will then provide a best fit to the full equations. Define a quantity, say E, that is an integral of the modulus of the right-hand side squared,

$$E = \int |F_0 + G \times F_1|^2 \,\mathrm{d}\eta. \tag{11}$$

Differentiating with respect to G and equating to zero gives

$$\tilde{G}_{best} = -\frac{\displaystyle\int F_1 \tilde{F}_0 \,\mathrm{d}\eta}{\displaystyle\int F_1 \tilde{F}_1 \,\mathrm{d}\eta}. \tag{12}$$

This value of G will provide an overall solution that fits the perturbation equations as closely as possible. The ODE used for the basic solution can be the Orr–Sommerfeld equation, or a modified version of it. The above solution and evaluation of the amplitude scaling is quite general and can be expected to provide approximate predictions of the downstream disturbance behaviour in a developing boundary layer. The solution is the best fit that one can obtain with the constraints imposed. There are, however, further simplifications that can be made if the main concern is the determination of the real part of G that controls the correction to the amplification rate. It turns out that by moving some of elements in F_0 across to the left-hand side of the equation to create a modified version of the Orr–Sommerfeld equation a more tractable form of scaling integral relation is created. This manoeuvre will simplify the subsequence analysis without any loss of precision. The modified equation is

$$\begin{aligned}(\alpha f' - \varpi\xi^{1/2})(\phi'' - \alpha^2\phi) - \alpha f'''\phi + \frac{\mathrm{i}}{R^{1/2}\,\xi^{1/2}}\Bigg\{&(\phi'''' - 2\alpha^2\phi'' + \alpha^4\phi) \\ &- \left[(\varpi\xi^{1/2} - 3\alpha f')\left(\frac{\xi\,\mathrm{d}\alpha}{\mathrm{d}\xi} + \frac{\alpha}{2}\right) + \alpha\varpi\xi^{1/2}\right]\phi \\ &+ (\alpha\varpi\xi^{1/2}\eta - \alpha^2\eta f' - \tfrac{1}{2}\alpha^2 f + \tfrac{1}{2}f'')\phi' + f'\phi''' + \tfrac{1}{2}f\phi'''\Bigg\} = 0.\end{aligned} \tag{13}$$

And F_{0new} and F_1 are

$$F_{0new} = (2\alpha\varpi\xi^{1/2} - 3\alpha^2 f' - f''')\frac{\xi\partial\phi}{\partial\xi} + f'\frac{\xi\partial\phi''}{\partial\xi} + \left[(2\varpi\xi^{1/2} - 6\alpha f')\frac{\xi\,\mathrm{d}\alpha}{\mathrm{d}\xi} + \alpha\varpi\xi^{1/2}\right]\phi, \tag{14a}$$

$$F_1 = (2\alpha\varpi\xi^{1/2} - 3\alpha^2 f' - f''')\phi + f'\phi''. \tag{14b}$$

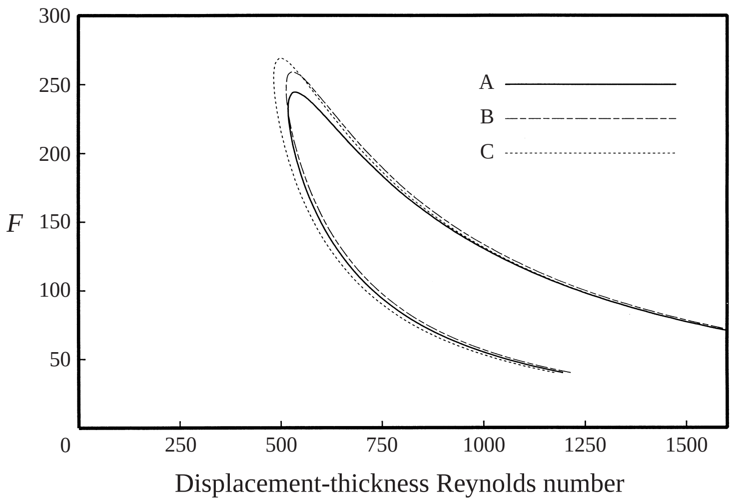

FIGURE 1. Neutral amplification plots: A, parallel flow approximation; B, adjoint method; C, current scheme.

And therefore

$$F_{0new} = \frac{\xi \partial F_1}{\partial \xi}. \tag{15}$$

With these modified equations

$$G_{real} = -\frac{\xi \dfrac{\mathrm{d}}{\mathrm{d}\xi} \displaystyle\int |F_1|^2 \,\mathrm{d}\eta}{2 \displaystyle\int |F_1|^2 \,\mathrm{d}\eta}. \tag{16}$$

This can be integrated with respect to ξ to provide an expression for the amplitude scaling of the solution of the ODE to account for boundary layer development with downstream distance:

$$A_{real}^2 = \frac{1}{\displaystyle\int |F_1|^2 \,\mathrm{d}\eta.} \tag{17}$$

At any specified distance from the boundary the streamwise amplification of a measurable velocity component is then

$$\left(\frac{\xi\,\mathrm{d}u}{u\,\mathrm{d}\xi}\right)_{real} = -\xi^{1/2} R^{1/2} \alpha_i + G_{real} + \left(\frac{\xi\,\mathrm{d}\phi'}{\phi'\,\mathrm{d}\xi}\right)_{real} - \tfrac{1}{2}. \tag{18}$$

4. Calculation

Equation (13) was solved by a shooting method using a simple purification scheme to prevent the divergent root from contaminating the result. This equation is more complex than the Orr–Sommerfeld equation because of the term involving the derivative of wavenumber with downstream position. By carrying out two simultaneous evaluations for close streamwise locations the term can be incorporated in the solution. This term is quite small and had only a minor influence on the final result; nevertheless its presence affected the convergence slightly as discussed in the introduction. The resulting neutral loop is shown on figure 1 together with that arising from the adjoint scheme.

5. Discussion

The adjoint method of solution generates a series of terms in inverse powers of $R^{1/2}$, where the Reynolds number is taken to be large. The leading term of this expansion is provided by an amplitude scaling as a function of streamwise distance multiplied by the solutions given by local parallel-flow predictions from the Orr–Sommerfeld equation. At infinite Reynolds number the leading term is dominant, and unless the series of terms has some unexpected problem it will also describe the wave behaviour at the large finite Reynolds numbers that are encountered in transition studies. The leading term satisfies the full linearized equations to order $R^{-1/2}$. Higher terms of the series absorb the remainder terms ignored in previous evaluations. Therefore, provided the series behaves reasonably, prediction could be made by summing the series. It turns out that for most boundary layer work it is only necessary to use the leading term to make predictions of wave growth with good accuracy. The present approach accepts the fact that only one term is required to adequately describe the wave evolution in a growing boundary layer and again considers a solution in the form of a scaled quasi-parallel flow prediction. But here the scaling function is determined so that the approximate solution satisfies the full linearized equations as closely as possible. This solution will locally fit the equations more closely than the leading term of the series produced by the adjoint method. If estimates of wave growth are required over large distances it may be that the adjoint methods will give more accurate results, but if the amplification part is small it may be that the present approach is more appropriate. The simplification made by re-arranging terms so that the modulus of the scaling equation can be integrated exactly is particularly helpful. Not only does this provide a conservation function, that one might intuitively have expected to occur in this type of problem, but it makes the calculation of wave amplification more straightforward.

The final form of approximation implies some conservation law to define the scaling, but the quantity being conserved here does not appear to be an obvious one. The growth of any physical quantity requires the calculation of the terms in (14*b*). The only quantity that can be given by an ODE without scaling is $\int |F_1|^2\,\mathrm{d}\eta$, but this does not appear to be linked to any measurable quantity nor is it a useful parameter in transition prediction.

The three neutral loops on figure 1 show that although the new method makes some corrections in the direction of the adjoint method the predictions are not precisely the same as one another. The method may, nevertheless, be a useful way of estimating non-parallel effects.

6. Conclusions

The present formulation provides a method of obtaining approximate solutions of a linear disturbance wave travelling downstream in a developing boundary layer. The method requires the solution of an ODE to obtain the eigenvalue and the eigenfunction at various streamwise locations. The scaling normalization is then given by an integral through the boundary layer of a simple function defined in terms of the eigensolutions. This is essentially much simpler to implement than methods that also require the adjoint function. The integral scaling enables growth corrections to be made between stations far apart without having to evaluate the corrections at all intermediate locations.

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